

A C^1 GENERIC CONDITION FOR EXISTENCE OF SYMBOLIC EXTENSIONS OF VOLUME PRESERVING DIFFEOMORPHISMS

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ABSTRACT. We prove that a C^1 -generic volume preserving diffeomorphism has a symbolic extension if and only if this diffeomorphism is partial hyperbolic. This result is obtained by means of good dichotomies. In particular, we prove Bonatti's conjecture in the volume preserving scenario. More precisely, in the complement of Anosov diffeomorphisms we have densely robust heterodimensional cycles.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

A system (X, f) has a *symbolic extension*, if there exist a subshift (Y, σ) , which is a closed, shift invariant subset of a full shift over a finite alphabet, and a surjective continuous map $\pi : Y \rightarrow X$ such that $\pi \circ \sigma = f \circ \pi$. In this case, (Y, σ) is called an *extension* of (X, f) and (X, f) a *factor* of (Y, σ) .

One way to measure the complexity of a system (X, f) could be by means of the topological entropy. Hence, if a system has a symbolic extension its complexity is bounded above by the complexity of a subshift. However, this system may contain additional information. The *symbolic extension entropy* of the system is the infimum of the topological entropy of all symbolic extensions of the system. And note that the topological entropy of a system is less than or equal to the symbolic extension entropy. The difference between these functions is called *residual entropy* and represents how entropy is hidden at finer and finer scales.

A symbolic extension of (X, f) is a *principal extension* if the map π is such that $h_\nu(\sigma) = h_{\pi_*\nu}(f)$ for every σ -invariant measure $\nu \in \mathcal{M}(\sigma|Y)$, where $h_\nu(\sigma)$ is the metric entropy of σ with respect to ν . Note, the residual entropy is zero if the system has a principal extension.

Let M be a Riemannian, connected and compact manifold. A diffeomorphism $f : M \rightarrow M$ is *asymptotically h -expansive* if

$$\lim_{\varepsilon \rightarrow 0} \sup_{x \in M} h(f|B_\infty(x, \varepsilon)) = 0,$$

where $B_\infty(x, \varepsilon) = \{y \in M; d(f^j(x), f^j(y)) < \varepsilon \text{ for every } j \in \mathbb{N}\}$, and $h(\cdot)$ denotes the topological entropy. If there exists ε_0 such that $\sup_{x \in M} h(f|B_\infty(x, \varepsilon)) = 0$ for every $0 < \varepsilon < \varepsilon_0$, then f is called *h -expansive*.

Boyle, D. Fiebig, U. Fiebig [9] using entropy structure showed that asymptotically h -expansive diffeomorphisms have principal extension. Hence, if a diffeomorphism has no symbolic extension, it should have entropy hidden no matter how thin is the scale. In other words, should there exist invariant subsets contained in balls with diameters arbitrary small having positive topological entropy. In some sense, this is in much the same way as the phenomena of coexistence of infinitely many horseshoes.

By a result of Buzzi [12], every smooth diffeomorphism over a compact manifold is asymptotically h -expansive, then it has a principal extension. A conjecture of Downarowicz and Newhouse in [20] asserts that every C^r -diffeomorphism ($r \geq 2$) has a symbolic extension. This conjecture was solved for surface diffeomorphisms by Burguet [10]. Also, we would like to remark that recently Burguet and Fisher [11] extended this result to higher dimensions proving that every C^2 partially hyperbolic diffeomorphisms with a 2-dimensional center bundle has a symbolic extension.

Hence, it seems natural to try to relate the existence of symbolic extensions to the differential structure of a diffeomorphism. For instance, using shadowing we can easily find a symbolic extension for Anosov diffeomorphisms. Diaz, Fisher, Pacifico and Vieitez [19] showed that every C^1 partial hyperbolic diffeomorphism is h -expansive if we define partial hyperbolic diffeomorphisms as in [17]. Hence it has a principal extension. See also [18]. A diffeomorphism f exhibits a *homoclinic tangency* if there exists a hyperbolic periodic point p of f having a non transversal homoclinic point. Thus, Liao, Viana and Yang proved that if a diffeomorphism is not approximated by one exhibiting homoclinic tangency, then it is also h -expansive.

On the other hand, we can consider a problem about the existence of diffeomorphisms that has no symbolic extensions. We can note that such diffeomorphisms have a rich dynamics, since they are not asymptotically h -expansive. Moreover, if the conjecture of Downarowicz and Newhouse is right such diffeomorphisms can not be C^2 .

In the symplectic scenario the author with Tahzibi [14] extended a result of Downarowicz and Newhouse [20], proving that C^1 -generically either a symplectic diffeomorphism is Anosov or has no symbolic extensions. That is, in the symplectic setting we have a large set of diffeomorphisms having no symbolic extensions.

The aim of this paper is to obtain similar results in the conservative case.

We denote by $\text{Diff}_m^1(M)$ the set of C^1 volume preserving diffeomorphisms over M . Here, as in [17], a f -invariant subset Λ is *partial hyperbolic* if there exists a continuous Df -invariant splitting $T_\Lambda M = E^s \oplus E_1^c \oplus \dots \oplus E_k^c \oplus E^u$, such that each center bundle E_i^c is one-dimensional, and there exist constants $m \in \mathbb{N}$, $0 < \lambda < 1$ such that for every $x \in M$:

$$\begin{aligned} -\|Df^m(v)\| &\leq 1/2 \text{ for each unitary } v \in E^s \text{ (one says } E^s \text{ is (uniformly) contracted.)}, \\ -\|Df^{-m}(v)\| &\leq 1/2 \text{ for each unitary } v \in E^u \text{ (one says } E^u \text{ is (uniformly) expanded.)}, \\ -\|Df_x^m(u)\| &\leq 1/2\|Df_x^m(v)\|, \text{ for each } x \in \Lambda, \text{ each } i = 0, \dots, k \text{ and each unitary} \\ &\text{vectors } u \in E^s \oplus \dots \oplus E_i^c, v \in E_{i+1}^c \oplus \dots \oplus E^u \text{ in } T_x M. \end{aligned}$$

If all center bundles are trivial, then Λ is called a *hyperbolic set*. We say that a volume preserving diffeomorphism $f : M \rightarrow M$ is *partial hyperbolic* if M is a partial hyperbolic set. If M is a hyperbolic set then we say that f is an Anosov diffeomorphism.

The main result of this paper is the following:

Theorem A. *There is a residual subset $\mathcal{R} \subset \text{Diff}_m^1(M)$ ($\dim M \geq 3$) such that if $f \in \mathcal{R}$ is a non partial hyperbolic diffeomorphism then f has no symbolic extension.*

Remark 1.1. In dimension two, the previous theorem follows from Downarowicz and Newhouse's result [20].

Now, Theorem A and the result of Diaz, Fisher, Pacífico and Vieitez [19] provide a generic intrinsic characterization of the existence of symbolic extensions in the volume preserving scenario.

Theorem B. *There exists a residual subset $\mathcal{R} \subset \text{Diff}_m^1(M)$ ($\dim M \geq 3$) such that a diffeomorphism $f \in \mathcal{R}$ has a symbolic extension if and only if it is partial hyperbolic. In particular, if $f \in \mathcal{R}$ has a symbolic extension then it has a principal extension.*

A directly consequence of this result is the following.

Corollary C. *If a C^1 generic volume preserving diffeomorphism f is conjugated to a C^∞ diffeomorphism, then f is partial hyperbolic.*

In the articles [20] and [14] the main tool to obtain their results is the existence of an “abundance” of diffeomorphisms exhibiting homoclinic tangency in the complement of Anosov diffeomorphisms, since they are in the symplectic scenario. Hence, it is somewhat folklore the relation between robustness of homoclinic tangency and non existence of symbolic extensions.

A diffeomorphism $f \in \text{Diff}^1(M)$ (resp. $\text{Diff}_m^1(M)$) exhibits a C^1 *robust homoclinic tangency* if there exist a hyperbolic basic set Λ of f and a small neighborhood $\mathcal{U} \subset \text{Diff}^1(M)$ (resp. $\text{Diff}_m^1(M)$) of f such that $W^s(\Lambda(g))$ has a non transversal intersection with $W^u(\Lambda(g))$ for every $g \in \mathcal{U}$. Where $\Lambda(g)$ is the continuation of Λ for g .

Proposition D. *If $f \in \text{Diff}^1(M)$ (resp. $\text{Diff}_m^1(M)$) exhibits a C^1 robust homoclinic tangency, then there exists a residual subset \mathcal{R} in some neighborhood of f in $\text{Diff}^1(M)$ (resp. $\text{Diff}_m^1(M)$), such that every $g \in \mathcal{R}$ has no symbolic extensions.*

As a consequence of this result we will obtain Theorem A. For that, we should investigate the relation between robustness of homoclinic tangency and partial hyperbolicity. More general, we should investigate the existence of good dichotomies.

Recently, Crovisier, Sambarino and Yang [17] proved that diffeomorphisms in $\text{Diff}^1(M)$ far from diffeomorphisms exhibiting homoclinic tangency, are approximated by partial hyperbolic diffeomorphisms. There they affirm that this result is also true in the conservative setting. Just for sake of completeness we will state it here and a sketch of the proof will appear inside the proof of Lemma 3.5, see Remark 3.6.

Proposition 1.2. Any diffeomorphism f can be approximated in $\text{Diff}_m^1(M)$ by diffeomorphisms which exhibit a homoclinic tangency or by partially hyperbolic diffeomorphisms.

We define the *index* of a hyperbolic periodic point p as the dimension of its stable manifold and we denote it by $\text{ind } p$. A diffeomorphism f exhibits a *heterodimensional cycle* if there exist hyperbolic periodic points p and q with different indices such that $W^s(p) \cap W^u(q)$ and $W^u(p) \cap W^s(q)$ are non empty intersections.

One open problem about dichotomies is Palis’s conjecture, which says that densely in $\text{Diff}^r(M)$ ($r \geq 1$) either a diffeomorphism is hyperbolic, or exhibits a homoclinic tangency, or exhibits a heterodimensional cycle. Palis’s conjecture was proved for C^1 surface diffeomorphisms by Pujals and Sambarino [32], and recently Crovisier and Pujals proved a remarkable result in this direction, by means of essential hyperbolicity, for this result see [16]. For symplectic and volume preserving

diffeomorphisms, there are also complete proofs for Palis's conjecture, see [29], [2] and [15]. We would like to remark that in the volume preserving case what was proved, in fact, is that in the lack of hyperbolicity there are densely diffeomorphisms exhibiting heterodimensional cycles. Hence, our next theorem is a generalization of this result.

A diffeomorphism $f \in \text{Diff}^1(M)$ (resp. $\text{Diff}_m^1(M)$) exhibits a C^1 *robust heterodimensional cycle* if there exist a hyperbolic basic set Λ and a hyperbolic periodic point p of f , with $\text{ind } \Lambda \neq \text{ind } p$, such that $\Lambda(g)$ and $p(g)$ exhibits a heterodimensional cycle, i.e., $W^s(\Lambda(g)) \cap W^u(p(g))$ and $W^u(\Lambda(g)) \cap W^s(p(g))$ are non empty intersections, for every diffeomorphism g in a neighborhood of f in $\text{Diff}^1(M)$ (resp. $\text{Diff}_m^1(M)$).

Theorem E. *There is an open and dense subset $\mathcal{A} \subset \text{Diff}_m^1(M)$ ($\dim M \geq 3$), such that if $f \in \mathcal{A}$ is a non Anosov diffeomorphism, then f exhibits a C^1 robust heterodimensional cycle.*

It is worth to pointing out that the previous theorem is in fact Bonatti's conjecture restrict to the volume preserving scenario. See [7].

In order to prove Theorem E, we use the so known blenders. Bonatti and Diaz in [7] developed a way to obtain diffeomorphisms exhibiting robust homoclinic tangencies from blenders. In this paper, we develop their technics in the conservative case to prove the following result:

Theorem F. *There is an open and dense subset $\mathcal{D} \subset \text{Diff}_m^1(M)$ ($\dim M \geq 3$) such that if $f \in \mathcal{D}$ is a non partial hyperbolic diffeomorphism then f exhibits a robust homoclinic tangency.*

Note, Theorem A is a directly consequence of the previous theorem and Proposition D.

This paper is organized as follows: in the second section we recall some useful perturbation results, and we show how to build a special kind of blender in the volume preserving setting. This special kind of blender is a blender horseshoe introduced in [7]. In this section we will prove Theorem E, too. In Section 3, we prove Theorem F, and finally, in Section 4, we prove Proposition D and Theorem A.

2. A BLENDER HORSESHOE AND SOME PERTURBATION RESULTS

Let f be a C^1 diffeomorphism on M . A hyperbolic transitive set Γ of f with $\dim W^u(\Gamma, f) = k \geq 2$ is a *cu-blender* if there exist a C^1 -neighborhood \mathcal{U} of f and a C^1 -open set \mathcal{D} of embeddings of $(k-1)$ -dimensional disks D into M such that, for every diffeomorphism $g \in \mathcal{U}$, every disk $D \in \mathcal{D}$ intersects the local stable manifold $W_{loc}^s(\Gamma(g))$, where $\Gamma(g)$ is the continuation of the hyperbolic set Γ for g . \mathcal{D} is called the superposition region of the blender. Similarly, we can define a *cs-blender* with stable manifold replaced by unstable manifold. The above definition was given in [7]. We would like to remark, that for a *cu-blender*, these $(k-1)$ -dimensional disks are usually *uu*-disks. See Remark 2.12. We refer the reader to [6] and [7] for more details about the geometry structure of this amazing set.

The main result of this section is the following.

Proposition 2.1. *If $f \in \text{Diff}_m^1(M)$ has two hyperbolic periodic points p_1 and p_2 of different indices, say i and $i+j$, respectively, then for any neighborhood*

$\mathcal{U} \subset \text{Diff}_m^1(M)$ of f and any $i \leq k \leq i + j - 1$ there exists a diffeomorphism $g \in \mathcal{U}$ having a cu -blender horseshoe Γ with index k .

Remark 2.2. A cu -blender horseshoe is a special kind of a cu -blender set, which will be defined in the proof of Proposition 2.1.

Remark 2.3. A similar result still holds for cs -blenders. More precisely, if f is under the same hypotheses of Proposition 2.1, then in any neighborhood of f there exists a diffeomorphism g having a cs -blender horseshoe of index k , for any $i+1 \leq k \leq i+j$.

We also remark that F. Hertz, M. Hertz, Tahzibi and Ures have already showed in [23] the existence of blenders in the conservative scenario, after some perturbation, if the initial diffeomorphism is under the same hypotheses of Proposition 2.1. However, we observe that here we are interested in a special kind of blenders. Moreover, we emphasize that our methods to prove Proposition 2.1 is different from their.

We will recall now some useful perturbation results.

The first one is a Pasting lemma of Arbieto and Matheus [3].

Theorem 2.4 (Pasting lemma). *If f is a C^2 volume preserving diffeomorphism over M , and $x \in M$, then for every $\varepsilon > 0$ there exists a C^1 volume preserving diffeomorphism $g \in C^1$ close to f such that for a small neighborhood $U \supset V$ of x , $g|_{U^c} = f$ and $g|_V = Df(x)$ (in local coordinates).*

Remark 2.5. If f is C^∞ then g could be taken C^∞ , too.

A directly consequence of pasting lemma is a conservative version of Franks lemma, see [25].

Lemma 2.6 (Franks lemma). *Let $f \in \text{Diff}_m^1(M)$ and \mathcal{U} be a C^1 neighborhood of f in $\text{Diff}_m^1(M)$. Then, there exists a smaller neighborhood $\mathcal{U}_0 \subset \mathcal{U}$ of f and $\delta > 0$ such that if $g \in \mathcal{U}_0(f)$, $S = \{x_1, \dots, x_m\} \subset M$ be any finite piece of orbit and $\{L_i : T_{x_i}M \rightarrow T_{x_{i+1}}M\}_{i=1}^m$ conservative linear maps satisfying $\|L_i - Dg(x_i)\| \leq \delta$ for every $i = 1, \dots, m$, then for any small fixed neighborhood V of S there exist $h \in \mathcal{U}(f)$ in the same class of differentiability of g , such that $h = g$ in V^c , moreover $h(x_i) = g(x_i)$ and $Dh(x_i) = L_i$.*

The next result is a connecting lemma of Hayashi [22]. A conservative version was proved by Wen and Xia [37].

Theorem 2.7 (C^1 -connecting lemma). *Let $f \in \text{Diff}_m^1(M)$ and p_1, p_2 hyperbolic periodic points of f , such that there exist sequences $y_n \in M$ and positive integers k_n such that:*

- $y_n \rightarrow y \in W_{loc}^u(p_1, f)$, $y \neq p_1$; and
- $f^{k_n}(y_n) \rightarrow x \in W_{loc}^s(p_2, f)$, $x \neq p_2$.

Then, there exists a C^1 volume preserving diffeomorphism g C^1 -close to f such that $W^u(p_1, g)$ and $W^s(p_2, g)$ have a non empty intersection close to y .

The following technical result will be needed in the proof of Proposition 2.1.

Lemma 2.8. *If $f \in \text{Diff}_m^1(M)$ has two hyperbolic periodic points p_1 and p_2 of different indices, say i and $i + j$, respectively, then for any neighborhood $\mathcal{U} \subset \text{Diff}_m^1(M)$ of f , $i \leq k \leq i + j - 1$ and $\varepsilon > 0$ there exists a diffeomorphism $g \in \mathcal{U}$ with a hyperbolic periodic point p , such that p has index k , $Dg^{\tau(p,g)}$ has only real*

eigenvalues with multiplicity one, say $\lambda_1 < \dots < \lambda_d$, and moreover $|\lambda_{k+1} - 1| < \varepsilon$. Where $\tau(p, g)$ denotes the period of p for g .

This lemma follows by the same method as in Proposition 3.2 in [2]. However, provided this method will be useful later we will give a sketch of the proof.

Before we prove the above lemma, let us recall some definitions.

Recall, two hyperbolic periodic points p and q , having the same index are *homoclinically related* if there exist a transversal intersection between $W^s(p, f)$ and $W^u(q, f)$, and $W^u(p, f)$ and $W^s(q, f)$. We denote by $H(p, f)$ the closure of the hyperbolic periodic points homoclinically related to p , which is called the *homoclinic class* of p . Similarly, we can define when hyperbolic periodic points and hyperbolic sets are homoclinically related.

A continuous Df -invariant splitting $T_\Lambda M = E_1 \oplus \dots \oplus E_k$ for a f -invariant subset Λ is *dominated* if the third condition in the partial hyperbolic definition is satisfied.

For abbreviation, sometimes we use expressions like "after a perturbation", or "there exists a diffeomorphism C^1 -close", which means that these perturbations could be done so small as we want.

Proof of Lemma 2.8:

By a result of Xia [35], a generic volume preserving diffeomorphism has all homoclinic classes non trivial. Thus, after a perturbation, we can assume $H(p_1, f)$ and $H(p_2, f)$ are non trivial. Now, by results of Bonatti, Diaz and Pujals [8], and Franks lemma we can perturb f to f_1 in order to obtain \tilde{p}_1 and \tilde{p}_2 hyperbolic periodic points homoclinically related to $p_1(f_1)$ and $p_2(f_1)$, respectively, such that $Df_1^{\tau(\tilde{p}_1, f_1)}(\tilde{p}_1)$ and $Df_1^{\tau(\tilde{p}_2, f_1)}(\tilde{p}_2)$ have only real eigenvalues with multiplicity one.

To simplify the notation we replace \tilde{p}_1 and \tilde{p}_2 by p_1 and p_2 , respectively. And moreover, we will continue to write p_1 and p_2 for their continuations.

In the sequence we perturb f_1 in order to find a diffeomorphism exhibiting a heterodimensional cycle between p_1 and p_2 . For that, we use a result of Bonatti and Crovisier [5]:

Proposition 2.9 (Bonatti and Crovisier). There exists a residual subset \mathcal{R} of $\text{Diff}_m^1(M)$ such that if $g \in \mathcal{R}$ then there exists a hyperbolic periodic point p of g such that $M = H(p, g)$. In particular, g is transitive.

After a perturbation, we can assume $f_1 \in \mathcal{R}$, i.e., f_1 is transitive. Then, using connecting lemma we can perturb f_1 to f_2 such that there is a non transversal intersection between $W^u(p_1, f_2)$ and $W^s(p_2, f_2)$. Since this intersection is robust, we can repeat the above process and perturb f_2 to f_3 such that $W^s(p_1, f_3)$ and $W^u(p_2, f_3)$ have also a non empty intersection, which implies that f_3 exhibits a heterodimensional cycle between p_1 and p_2 . Moreover, f_3 can be taken such that $Df_3^{\tau(p_1, f_3)}(p_1)$ and $Df_3^{\tau(p_2, f_3)}(p_2)$ have only real eigenvalues with multiplicity one.

Let $x \in W^s(p_1, f_3) \cap W^u(p_2, f_3)$ and $y \in W^u(p_1, f_3) \cap W^s(p_2, f_3)$ be two heteroclinic points of the cycle. Recall $\text{ind } p_1 = i$ and $\text{ind } p_2 = i + j$. Without loss of generality we can assume y is a transversal heteroclinic point, and x is a quasi-transversal heteroclinic point, i.e., $T_x W^s(p_1, f_3) \cap T_x W^u(p_2, f_3) = \{0\}$. By the regularization result of Ávila [4] (which says we can suppose f_3 be C^∞) and pasting lemma, we can linearize the diffeomorphism in a small neighborhood U_{p_1} and U_{p_2} of p_1 and p_2 , respectively. More precisely, we can assume f_3 is equal to $Df_3(p_1)$ and $Df_3(p_2)$ (in local coordinates) in the neighborhoods U_{p_1} and U_{p_2} , respectively.

For simplicity of notation, in the reminder of this proof we assume that p_1 and p_2 are fixed points, and we will look at U_{p_1} and U_{p_2} in local coordinates. Since $Df_3(p_1)$ and $Df_3(p_2)$ have only real eigenvalues with multiplicity one, we can find a decomposition of \mathbb{R}^d by eigenspaces of $Df_3(p_1)$ (resp. $Df_3(p_2)$), which we denote by $E_{1,p_1} \oplus \dots \oplus E_{d,p_1}$ (resp. $E_{p_2} \oplus \dots \oplus E_{d,p_2}$). We set λ_k (resp. σ_k), $k = 1, \dots, d$, the eigenvalue of $Df_3(p_1)|_{E_{k,p_1}}$ (resp. $Df_3(p_2)|_{E_{k,p_2}}$). We can also suppose the eigenvalues are in an increase order.

In order to be more precise, we will make the following assumptions, we consider $E_{i,p_1}(\cdot)$ the extension of the direction E_{i,p_1} in the neighborhood U_{p_1} , the same for $E_{i,p_2}(\cdot)$. We remark these decompositions are all dominated splittings, indeed.

$\text{ind } p_1 = i$, it follows that the stable and unstable directions of p_1 are $E_{p_1}^s = E_{1,p_1}(p_1) \oplus \dots \oplus E_{i,p_1}(p_1)$ and $E_{p_1}^u = E_{i+1,p_1}(p_1) \oplus \dots \oplus E_{d,p_1}(p_1)$, respectively. Similarly, the stable and unstable directions of p_2 are $E_{p_2}^s = E_{1,p_2}(p_2) \oplus \dots \oplus E_{i+j,p_2}(p_2)$ and $E_{p_2}^u = E_{i+j+1,p_2} \oplus \dots \oplus E_{d,p_2}$, respectively, since $\text{ind } p_2 = i + j$. Moreover, by the choice of f_3 , $W_{loc}^s(p_1, f_3) = E_{p_1}^s \cap U_{p_1}$, $W_{loc}^u(p_1, f_3) = E_{p_1}^u \cap U_{p_1}$, $W_{loc}^s(p_2, f_3) = E_{p_2}^s \cap U_{p_2}$ and $W_{loc}^u(p_2, f_3) = E_{p_2}^u \cap U_{p_2}$.

Claim: *There is a diffeomorphism f_4 C^1 -close to f_3 such that the f_3 -invariant subset $\Lambda = O(x) \cup O(y) \cup \{p_1, p_2\}$ still is f_4 -invariant and moreover has a dominated splitting by one dimensional sub-bundles.*

We define $E(y) := T_y(W^u(p_1, f_3) \cap W^s(p_2, f_3))$. Since y belongs to unstable manifold of p_1 , and $f_3|_{U_{p_1}} = Df_3(p_1)$, if n is large enough, it follows that $Df_3^{-n}(y)(E(y))$ is in $E_{i+1,p_1}(f_3^{-n}(y)) \oplus \dots \oplus E_{d,p_1}(f_3^{-n}(y))$. Moreover, by transversality we can assume that $Df_3^{-n}(y)(E(y)) \cap E_{i+j+1,p_1}(f_3^{-n}(y)) \oplus \dots \oplus E_{d,p_1}(f_3^{-n}(y)) = \{0\}$. Provided we have a dominated splitting in U_{p_1} , $Df_3^{-n}(y)(E(y))$ converges to $E_{i+1,p_1}(p_1) \oplus \dots \oplus E_{i+j,p_1}(p_1)$ when $n \rightarrow \infty$. Then, choosing n large enough and using Franks lemma, after a perturbation we can assume f_3 such that $Df_3^{-n}(y)(E(y)) = E_{i+1,p_1}(f_3^{-n}(y)) \oplus \dots \oplus E_{i+j,p_1}(f_3^{-n}(y))$. Note, the perturbation necessary here is local, and moreover keeps unchanged the orbit of y .

We now apply this argument again, considering the future orbit of y , to obtain a perturbation of f_3 such that we have also $Df_3^n(y)(E(y)) = E_{i+1,p_2}(f_3^n(y)) \oplus \dots \oplus E_{i+j,p_2}(f_3^n(y))$. This perturbation of f_3 which we continue denoting by the same letter has a Df_3 -invariant subbundle on $O(y) \cup \{p, q\}$ which we will denote by E , for convenience.

The λ -lemma says that $Df_3^m(f_3^{-n}(y))(E_{i+j+1,p_1}(f_3^{-n}(y)) \oplus \dots \oplus E_{d,p_1}(f_3^{-n}(y)))$ converges to $E_{i+j+1,p_2}(p_2) \oplus \dots \oplus E_{d,p_2}(p_2)$ if $m \rightarrow \infty$. Then by the same argument again we can perturb f_3 such that $Df_3^m(f_3^{-n}(y))(E_{i+j+1,p_1}(f_3^{-n}(y)) \oplus \dots \oplus E_{d,p_1}(f_3^{-n}(y))) = E_{i+j+1,p_2}(f_3^{m-n}(y)) \oplus \dots \oplus E_{d,p_2}(f_3^{m-n}(y))$, and the subbundle E is still Df -invariant. Replacing m and n by large positive integers if necessary, and applying once more the argument, f_3 could be assumed such that $Df_3^{-m}(f_3^{m-n}(y))(E_{1,p_2}(f_3^{m-n}(y)) \oplus \dots \oplus E_{i,p_2}(f_3^{m-n}(y))) = E_{1,p_1}(f^{-n}(y)) \oplus \dots \oplus E_{i,p_1}(f^{-n}(y))$.

Therefore, f_3 is such that there exists a Df_3 -invariant splitting over $O(y) \cup \{p_1, p_2\}$.

Moreover, if we repeat this process finitely many times inside each invariant sub-bundle, f_3 could be assumed such that

$$Df_3^{2n}(f_3^{-n}(y)(E_{k,p_1}(f_3^{-n}(y)))) = E_{k,p_2}(f_3^n(y)), \quad k = 0, \dots, d; \text{ for } n \text{ large enough.}$$

Finally, applying the above arguments, with y replaced by x , f_3 can also be chosen such that

$$Df_3^{2n}(f_3^{-n}(x))(E_{k,p_2}(f_3^{-n}(x))) = E_{k,p_1}(f_3^n(x)), \quad k = 0, \dots, d; \text{ for } n \text{ large enough,}$$

which finishes the proof of the claim, since this Df -invariant splitting is natural dominated.

We fix now an arbitrary $i \leq k \leq i+j-1$, and we consider the diffeomorphism f_4 given by the previous Claim. Using the heteroclinic points x and y , we can perform a perturbation of f_4 to obtain a periodic orbit in a small neighborhood of Λ , with arbitrary large period. In fact, this could be done such that this periodic orbit has as many points as we want in the neighborhoods U_{p_1} and U_{p_2} , being fixed the number of points outside these neighborhoods. Hence, by continuity of the dominated splitting over Λ , and since $\|Df_4|E_{k+1}(p_1)\| > 1$ and $\|Df_4|E_{k+1}(p_2)\| < 1$, it follows there exists a diffeomorphism f_5 C^1 -close to f_4 having a hyperbolic periodic point p in a neighborhood of $\Lambda(f_5)$ with index k and such that $Df_5^{\tau(p,f_5)}(p)$ has only real eigenvalues with multiplicity one. Moreover, this could be done such that $\|Df_5^{\tau(p,f_5)}|E_{k+1}(p)\|$ is so close to one as we want. For details we refer the reader to [2]. □

Proof Proposition 2.1 :

We fix an arbitrary $i \leq k \leq i+j-1$. By Lemma 2.8, after a perturbation, we can assume there exists a hyperbolic periodic point p of f such that p has index k , $Df^{\tau(p,f)}(p)$ has only real eigenvalues with multiplicity one, say $\lambda_1 < \dots < \lambda_d$, and moreover λ_{k+1} is so close to one as we want.

If E_{λ_t} is the corresponding eigenspace to λ_t , then we have on p a natural partially hyperbolic splitting $T_p M = E^s \oplus E^{cu} \oplus E^{uu}$, where $E^s = \cup_{1 \leq t \leq k} E_{\lambda_t}$ is the stable direction of dimension k , and the unstable direction is divided in two subspaces, $E^{cu} = E_{\lambda_{k+1}}$ (the center unstable direction), and $E^{uu} = \cup_{t > k+1} E_{\lambda_t}$ the strong unstable direction. By Hirsch, Pugh and Shub [24], the strong directions are integrable, which means here the existence of $W^{uu}(p, f)$, the *strong unstable manifold* of p , which varies continuously with respect to the diffeomorphism.

As in the proof of Lema 2.8, we can perturb f to a C^∞ diffeomorphism f_1 to obtain a intersection between the stable and strong unstable manifolds of $p(f_1)$, $W^s(p(f_1), f_1) \cap W^{uu}(p(f_1), f_1) \neq \emptyset$, and moreover such that $f_1^{\tau(p_1(f_1), f_1)} = Df_1^{\tau(p_1(f_1), f_1)}(p(f_1))$ (in local coordinates) in a neighborhood U of $p(f_1)$.

By abuse of notation, we write just p instead of $p(f_1)$. Also, since $\|Df^{\tau(p,f)}(p)|E^{cu}\|$ is so close to one as we want, after another perturbation, we can suppose $|\tilde{\lambda}_c| = \|Df_1^{\tau(p,f_1)}(p)|E^{cu}\| = 1$.

From now on, we look at U in local coordinates. Then, in U the local stable and strong unstable manifolds of p coincide with their directions, i.e., $W_{loc}^s(p, f_1) = E^s(p, f_1) \cap U$ and $W_{loc}^{uu}(p, f_1) = E^{uu}(p, f_1) \cap U$.

Let $x \in W^s(p, f_1) \cap W^{uu}(p, f_1)$ be a strong homoclinic point of p , which we can also assume be a quasi-transversal strong homoclinic point, i.e., $\dim(T_x W^s(p, f_1) + T_x W^{uu}(p, f_1)) = d-1$ since $\dim E^{cu}(p) = 1$. Hence, there exist positive integers n and m such that $f_1^n(x) = (x_0^s, 0, 0)$, $f_1^{-m}(x) = (0, 0, x_0^u) \in U$. Here, we are considering the natural extension to U of the partial hyperbolic splitting $T_p M =$

$E^s \oplus E^{cu} \oplus E^{uu}$. Also, without loss of generality we can suppose this decomposition orthogonal.

By the same method as in the Claim in the proof of Lemma 2.8, we can find a diffeomorphism f_2 C^1 -close to f_1 , such that shrinking U if necessary f_2 satisfies the following conditions:

- 1- $f_2^{\tau(p, f_2)} = Df_2^{\tau(p, f_2)}(p) = Df_1^{\tau(p, f_1)}(p)$ in U , keeping invariant the directions $E^j \cap U$, $j = s, cu, uu$;
- 2- x is still a strong homoclinic point of p , and moreover

$$Df_2^{mn}(f_2^{-m}(x))(E^j(f_2^{-m}(x))) = E^j(f^n(x)), \quad j = s, cu, uu.$$

f_2 is obtained through several finitely many perturbations of f_1 using Franks lemma, f_2 is in the same class of differentiability of f_1 , which implies f_2 is C^∞ . Hence, we can use pasting lemma in order to linearize f_2 in a segment of the orbit of x . More precisely, we can choose neighborhoods $U_m, U_n \subset U$ of $f_3^{-m}(x)$ and $f_3^n(x)$, respectively, and perturb f_2 to f_3 such that $f_3^{mn}(E^j(y) \cap U_m) = E^j(f^{mn}(y)) \cap U_n$, for every $y \in U_m$ and $j = s, cu, uu$. Using the fact that $f_2^{\tau(p, f_2)}$ is linear in U and $\tilde{\lambda}_c = 1$, replacing m and n with larger ones, and after once more perturbation, we can suppose f_3 satisfying

- 3- $f_3^{mn} : U_m \rightarrow U_n$ is an affine map. More precisely,

$$f_3^{mn}(x^s, x^c, x^u) = (x_0^s + A_s(x^s), \lambda_c x^c, A_u(x^u - x_0^u)),$$

where A_s is a linear contraction, A_u a linear expansion and $1 < \lambda_c < 1 + \epsilon$, for some small $\epsilon > 0$.

- 4- $E^s \oplus E^{uu}$ is invariant for both maps $f_3^{\tau(p, f_3)}|_U$ and $f_3^{mn}|_{U_m}$.

Hence, if $D \subset (E^s \oplus E^{uu}) \cap U$ is a small enough rectangle containing p and $f_3^n(x)$ in its interior, then $f_3^{l\tau(p, f_3)+mn}(D) \cap D$ has two non empty disjoint connected components for some l large enough. One of them containing p and another one $f_3^n(x)$, which we denote by \mathcal{A} and \mathcal{B} , respectively.

For simplicity of notation, we set $\tilde{F} = f_3^{l\tau(p, f_3)+mn}|_D$, $\mathbb{A} = \tilde{F}^{-1}(\mathcal{A})$ and $\mathbb{B} = \tilde{F}^{-1}(\mathcal{B})$. Note, \tilde{F} is a linear map on $\mathbb{A} \cup \mathbb{B}$, and the stable and strong unstable directions are \tilde{F} -invariant. Moreover, taking l larger if necessary $\tilde{F}|_{E^s}$ and $\tilde{F}^{-1}|_{E^{uu}}$ are contractions, for every point in $\mathbb{A} \cup \mathbb{B}$ and $\mathcal{A} \cup \mathcal{B}$, respectively. Hence, the maximal invariant set in D for \tilde{F} ,

$$\Sigma = \bigcap_{j \in \mathbb{Z}} \tilde{F}^j(D)$$

is a hyperbolic set conjugated to the full shift of two symbols. We denote by $q \in \mathbb{B}$ the other fixed point of \tilde{F} . Note, $E^s \oplus E^{uu}$ is the hyperbolic splitting over Σ .

Fixing any arbitrary small $\delta > 0$, we set $R = D \times [-\delta, \delta] \subset U$, and replace \mathbb{A} and \mathbb{B} by $\mathbb{A} \times [-\delta, \delta]$ and $\mathbb{B} \times [-\delta, \delta]$, respectively. Taking δ smaller, $F := f_3^{l\tau(p, f_3)+mn}|_{\mathbb{A} \cup \mathbb{B}}$ is then well defined. Moreover, taking the center coordinate as the last one, we have

$$F(x^s, x^u, x^c) = (\tilde{F}(x^s, x^u), \lambda_c x^c).$$

Since $\lambda_c > 1$, it follows that $\Lambda_0 = \Sigma \times 0$ is the maximal F -invariant set in R . Also, provided E^s , E^{cu} and E^{uu} are F -invariant, we have a natural partial hyperbolic splitting on Λ_0 . In particular, Λ_0 is a hyperbolic set with index k since $\|F|_{E^{cu}}\| > 1$.

After a coordinate change, we can suppose $R = [-1, 1]^s \times [-1, 1] \times [-1, 1]^u$ in local coordinates, and $p = (0, 0, 0)$ in this chart.

For every $t > 0$ small enough, using pasting lemma we can find a perturbation h_t of the identity map such that $h_t(x^s, x^c, x^u) = (x^s, x^c - t, x^u)$ for every point in U_n and $h_t = Id$ outside a small neighborhood of U_n . We define $f_t = h_t \circ f_3$, which is C^1 -close to f_3 .

Shrinking U_n if necessary, the above perturbation f_t in terms of F_t is the following

$$\begin{aligned} 1- & F_t = F, & \text{if } x \in \mathbb{A} \\ 2- & F_t = F + (0, -t, 0), & \text{if } x \in \mathbb{B}. \end{aligned}$$

Provided t is small, the maximal F_t -invariant set Λ_t in R is the continuation of the hyperbolic set Λ_0 of F , hence Λ_t is also hyperbolic. Moreover, note $E^s \oplus E^{cu} \oplus E^{uu}$ is still the hyperbolic splitting on Λ_t , and p is still a hyperbolic fixed point of F_t . We denote by q_t the continuation of the hyperbolic fixed point q of F .

This set Λ_t is defined as a *cu-blender horseshoe*.

In the sequence, we will describe some properties of Λ_t which characterize, in fact, a cu-blender horseshoe.

For $\alpha \in (0, 1)$ we denote by C_α^s and C_α^{uu} the following cone-fields in R :

$$\begin{aligned} C_\alpha^s(x) &= \{v = (v^s, v^c, v^u) \in E^s \oplus E^{cu} \oplus E^{uu} = T_x M; \|v^c + v^u\| \leq \alpha \|v^s\|\} \\ C_\alpha^{uu}(x) &= \{v = (v^s, v^c, v^u) \in E^s \oplus E^{cu} \oplus E^{uu} = T_x M; \|v^s + v^c\| \leq \alpha \|v^u\|\}. \end{aligned}$$

We say that a disk Δ of dimension s contained in R is a *s-disk* if

- it is tangent to C_α^s , i.e., $T_x \Delta \subset C_\alpha^s(x)$ for all $x \in \Delta$, and
- its boundary $\partial \Delta$ is contained in $\{-1, 1\}^s \times [-1, 1] \times [-1, 1]^u$.

On the other hand, a disk Υ of dimension u is a *uu-disk* if

- it is tangent to C_α^{uu} , i.e., $T_x \Upsilon \subset C_\alpha^{uu}(x)$ for all $x \in \Upsilon$, and
- its boundary $\partial \Upsilon$ is contained in $[-1, 1] \times [-1, 1] \times \{-1, 1\}^u$.

Remark 2.10. As expected, $W_{loc}^s(p, F_t) = [-1, 1]^s \times \{0\} \times \{0\}$ and $W_{loc}^s(q_t, F_t) = [-1, 1]^s \times \{t/\lambda_c(\lambda_c - 1)\} \times \{x_0^u(q)\}$ are natural s -disks, while $W_{loc}^{uu}(p, F_t) = \{0\} \times \{0\} \times [-1, 1]^u$ and $W_{loc}^{uu}(q_t, F_t) = \{x_0^s(q_t)\} \times \{t/\lambda_c(\lambda_c - 1)\} \times [-1, 1]$ are natural uu -disks.

Note, there are two different homotopy classes of uu -disks disjoint from $W^s(p, F_t)$. We say that an uu -disk is at the right of p if it belongs to the same homotopy class of $W^{uu}(q_t, F_t)$, and at the left otherwise. Similarly, we say that a uu -disk is at the left of q_t if it belongs to the same homotopy class of $W^{uu}(p, F_t)$, and at the right otherwise.

By this convention, if \mathcal{D} is an uu -disk, then one of the following is true:

- \mathcal{D} is at the left of p ;
- $\mathcal{D} \cap W^s(p, F_t) \neq \emptyset$;
- \mathcal{D} is at the right of q_t ;
- $\mathcal{D} \cap W^s(q_t, F_t) \neq \emptyset$;
- \mathcal{D} is at the right of p and at the left of q_t . In this case we say that the uu -disk is in between of p and q_t .

We fix now a very small $\alpha \in (0, 1)$ in the definition of the uu -disks, such that a uu -disk \mathcal{D} is C^1 -close to E^{uu} . If we define $F_{\mathbb{A}}(\mathcal{D}) = F_t(\mathbb{A} \cap \mathcal{D})$ and $F_{\mathbb{B}}(\mathcal{D}) = F_t(\mathbb{B} \cap \mathcal{D})$, then the following is true:

- 1- If \mathcal{D} is at the right of p (resp. q_t) then $F_{\mathbb{A}}(\mathcal{D})$ (resp. $F_{\mathbb{B}}(\mathcal{D})$) also is.
- 2- If \mathcal{D} is at the left of p (resp. q_t) then $F_{\mathbb{A}}(\mathcal{D})$ (resp. $F_{\mathbb{B}}(\mathcal{D})$) also is.
- 3- If \mathcal{D} is at the left of p then $F_{\mathbb{B}}(\mathcal{D})$ also is.

- 4- If \mathcal{D} is at the right of q_t then $F_{\mathbb{A}}(\mathcal{D})$ also is.
- 5- If \mathcal{D} is in between of p and q_t , then either $F_{\mathbb{A}}(\mathcal{D})$ or $F_{\mathbb{B}}(\mathcal{D})$ is in between of p and q_t .

Remark 2.11. The above properties are robust. More precisely, if g is a diffeomorphism C^1 -close to f_t , and if we denote by Λ_g the continuation of the the hyperbolic periodic set Λ_t of f_t , then $g^{l\tau(p(g),g)+mn}|_{\Lambda_g} = G|_{\Lambda_g}$ has the same properties of $F_t|_{\Lambda_t}$. Therefore, a *cu*-blender horseshoe set is robust.

Remark 2.12. Using iterated functions is possible to verify that every *uu*-disk in between of $W^s(p(g), g)$ and $W^s(q_t(g), g)$ intersects $W_{loc}^s(\Lambda_g, g)$. See Bonatti and Diaz [6]. In particular, the blender horseshoe Λ_g is in fact a *cu*-blender, where the *uu*-disks in between of $W^s(p(g), g)$ and $W^s(q_t(g), g)$ define its superposition region.

To see more properties about a blender horseshoe set we refer the reader to [7].

□

We prove now Theorem E.

Proof of Theorem E:

Let f be a non Anosov volume preserving diffeomorphism. By Theorem 1.1 in [2], there exists a diffeomorphism $f_1 \in \text{Diff}_m^1(M)$ C^1 -close to f having a non-hyperbolic periodic point p . After a bifurcation of p we can assume that f_1 has two hyperbolic periodic points of different indices, say p_1 and p_2 , with $\text{ind } p_1 = i$ and $\text{ind } p_2 = i + j$, $i, j > 0$.

By Proposition 2.1 we can find a volume preserving diffeomorphism f_2 C^1 -close to f_1 such that f_2 has a blender horseshoe Λ with index $i + j - 1$. We replace now p_1 by one of the two reference saddles of Λ .

As in the proof of Proposition 2.1, we can perturb f_2 to f_3 to obtain a heterodimensional cycle between p_1 and p_2 . Let z denote a point of non transversal intersection between $W^s(p_1, f_3)$ and $W^u(p_2, f_3)$, which we can assume to be a quasi transversal intersection. Provided the partial hyperbolic structure in the superposition region \mathbb{C} of the blender, replacing z by a positive iterated, the connected disc in $W^u(p_2, f_3) \cap \mathbb{C}$ containing z is in fact a *uu*-disk which is in between of the two reference saddles of Λ , as defined in the proof of Proposition 2.1. Note, this could be done such that $W^u(p_1, f_3) \cap W^s(p_2, f_3)$ has a transversal intersection.

Therefore, by properties of blenders, Remark 2.12, and continuity of the unstable manifold of p_2 , every volume preserving diffeomorphism g in a small neighborhood of f_3 has a heterodimensional cycle between $p_2(g)$ and $\Lambda(g)$.

□

3. ROBUSTNESS OF HOMOCLINIC TANGENCY

In this section we prove Theorem F. For that, it will be necessary to introduce folded submanifolds, introduced by Bonatti and Diaz in [7].

Definition 3.1. Let f be a diffeomorphism on M having a blender-horseshoe set Λ of index $u + 1$ with reference cube \mathbb{C} , reference saddles p and q , and $N \subset M$ be a submanifold of dimension $u + 1$. We say that N is *folded with respect to* Λ if the interior of N contains a sub-manifold $\mathcal{S} \subset \mathbb{C} \cap N$ of dimension $u + 1$, satisfying the following properties:

- There are $0 < \alpha' < \alpha$ and a family $(S_t)_{t \in [0,1]}$ of disks tangent to the cone field $C_{\alpha'}^{uu}$, depending continuously on t , such that $\mathcal{S} = \cup_{t \in [0,1]} S_t$. Here, α comes from the definition of a blender horseshoe, in particular S_t is a uu -disk;
- $\mathcal{S}_0 \cap W_{loc}^s(A)$ and $\mathcal{S}_1 \cap W_{loc}^s(A)$ are non empty transverse intersection points between N and $W_{loc}^s(A)$, where $A \in \{p, q\}$.
- for every $t \in (0, 1)$, the uu -disk S_t is in between of $W_{loc}^s(p)$ and $W_{loc}^s(q)$.

To emphasize the reference saddle A of the blender we have considered, we say a submanifold N is folded with respect to (Λ, A) .

Theorem 3.2 (Theorem 2, pg 18, [7]). *Let f be a C^r ($r \geq 1$) diffeomorphism over M , and $N \subset M$ be a folded submanifold with respect to a blender-horseshoe Λ of f . Then N and $W_{loc}^s(\Lambda)$ have a non empty C^r -robust non transversal intersection.*

To prove Theorem F the following results will also be needed.

First, Wen has proved in [36] a dichotomy between diffeomorphisms having a dominated splitting and diffeomorphisms exhibiting a homoclinic tangency in the space of C^1 diffeomorphisms. Using the pasting lemma, Liang, Liu and Sun [25] proved this dichotomy in the volume preserving scenario.

Proposition 3.3. [Theorem 1.3 in [25]] Let $f \in \text{Diff}_m^1(M)$, and p be a hyperbolic periodic point of f . Then, we have the following dichotomy:

- 1- Either the homoclinic class of p , $H(p, f)$, has a dominated splitting $TM = E \oplus F$, with $\dim E = \text{ind}(p)$, or
- 2- there exists a diffeomorphism g C^1 -close to f , exhibiting a homoclinic tangency for $p(g)$.

The following result is a conservative version of Theorem 1 in [1]. It may be proved using the same arguments as in Lemma 2.8, see [2] for details.

Proposition 3.4. There is a residual subset $\mathcal{R} \in \text{Diff}_m^1(M)$ of diffeomorphisms f such that, for every $f \in \mathcal{R}$ containing hyperbolic periodic points of indices i and j contains hyperbolic periodic points of index k for all $i \leq k \leq j$.

Proof of Theorem F:

Let f be a volume preserving diffeomorphism which is not approximated by a partial hyperbolic diffeomorphism in $\text{Diff}_m^1(M)$. In particular, f is a non Anosov diffeomorphism, and then after a perturbation if necessary as in the proof of Theorem E we can assume f has hyperbolic periodic points of different indices.

We set i and j the smallest and largest positive integers, respectively, such that every hyperbolic periodic point p of f has $i \leq \text{ind} p \leq j$. we are in the volume preserving scenario, after a perturbation, we can suppose this is still true for diffeomorphisms in a small neighborhood \mathcal{U} of f . More precisely, if $g \in \mathcal{U}$ and p is a hyperbolic periodic point of g then $i \leq \text{ind} p \leq j$.

By Proposition 3.4 we can assume f such that there are hyperbolic periodic points of index k , for every $i \leq k \leq j$. Hence, there are q_i, \dots, q_j hyperbolic periodic points of f of indices i, \dots, j , respectively. Also, by Proposition 2.1 we can find a diffeomorphism f_1 C^1 -close to f , which is not approximated by partial hyperbolic diffeomorphisms, such that there exist blender horseshoe subsets Λ_k with $\text{ind} \Lambda_k = k$, for every $k = i, \dots, j-1$. By Remark 2.3, we can also assume there is a cs -blender horseshoe Λ_j with $\text{ind} \Lambda_j = j$.

By Proposition 2.9 and a result of Carballo, Morales and Pacifico [13], we can also suppose that $H(q_i(f_1), f_1) = \dots = H(q_j(f_1), f_1) = M$, i.e., hyperbolic periodic points of every index are dense in the whole manifold M . We would like to note that although the result in [13] is in dissipative setting, provided it is a consequence of the connecting lemma, it is still true in the volume preserving scenario.

Lemma 3.5. *There exists $p \in \{q_i(f_1), \dots, q_j(f_1)\}$ and a diffeomorphism f_2 C^1 -close to f_1 such that f_2 exhibits a homoclinic tangency for $p(f_2)$.*

Proof. Suppose, contrary to our claim, that every diffeomorphism in a small neighborhood of f_1 exhibits no homoclinic tangency for any hyperbolic periodic points $q_i(f_1), \dots, q_j(f_1)$. Then, f_1 is in fact not approximated by diffeomorphisms exhibiting homoclinic tangency.

Hence, by Proposition 3.3 we should have dominated splittings $TM = E_k \oplus F_k$ for f_1 , with $\dim E_k = k$, for every $k = i, \dots, i + j$. Which implies we have a dominated splitting $TM = E_i \oplus E^1 \oplus \dots \oplus E^{j-i-1} \oplus E_j$, where $\dim E^k = 1$ for every $1 \leq k \leq j - i - 1$.

Since for diffeomorphisms in \mathcal{U} any hyperbolic periodic point p has $\text{ind } p \leq j$, it follows by the same method as in Lemma 2.1 in [31], that there exists $K > 0$, $m \in \mathbb{N}$ and $0 < \lambda < 1$ such that every hyperbolic periodic point p of a diffeomorphism $g \in \mathcal{U}$ with index j and sufficiently large period one has

$$(3.1) \quad \prod_{l=0}^k \left\| \prod_{r=0}^{m-1} Dg^{-1}|E_j(g^{-lm-r}(p)) \right\| \leq K\lambda^k, \text{ where } k = \left\lceil \frac{\tau(p, g)}{m} \right\rceil.$$

To obtain this, Potrie [31] uses a result for uniformly contracting sequences introduced by Mañé, Lemma II.5 in [27].

Now, by a known Mañé's argument also introduced in [27], we can find a positive integer n such that $\|Df_1^{-n}|E_j\|$ contracts for every $x \in M$. This argument consists in use Mañé's Ergodic Closing Lemma to obtain a hyperbolic periodic point of index j that doesn't satisfy equation 3.1, if $\|Df^{-n}|E_j\|$ no contracts for every $n > 0$. All of these arguments, including the results for uniformly contracting sequences are done for volume preserving diffeomorphisms in [2].

Similarly, we can prove that $\|Df_1^n|E_i\|$ contracts for a large enough positive integer n . Then f_1 is partial hyperbolic, which is a contradiction and then finishes the proof of lemma. \square

Remark 3.6. We point out that the above arguments give in particular a proof of Proposition 1.2.

Hence, let f_2 and p given by Lemma 3.5. After a perturbation, we can suppose p is one of the two reference saddles of the blender-horseshoe $\Lambda = \Lambda_k(f_2)$, if $\text{ind } p = k$.

By Proposition 3.2, the proof is completed by showing that:

Lemma 3.7. *There is a diffeomorphism g arbitrarily C^1 -close to f_2 such that $W^u(p(g))$ is a folded submanifold with respect to the continuation $\Lambda(g)$ of the blender-horseshoe Λ for g .*

This lemma is a volume preserving version of Lemma 4.9 in [7].

Proof. Let B denote a point of homoclinic tangency for $p(f_2)$, which we can suppose to be in \mathbb{C} , i.e., B is in the reference cube of the blender horseshoe Λ . As in the proof of Lemma 2.8, after a perturbation, we can assume $T_B W^s(p, f_2) \cap$

$T_B W^u(p, f_2) = E^{cu}(B)$ is the one dimensional center-unstable subspace. Hence, let $\mathbb{V} \subset T_B W^u(p, f_2)$ be such that $T_B W^u(p, f_2) = \mathbb{V} \oplus E^{cu}$.

Provided we have a partial hyperbolic splitting in \mathbb{C} , $Df^n(B)(\mathbb{V})$ converges to $E^{uu}(p)$, if n goes to infinity. Hence, if $U \subset W^u(p, f_2)$ is a small enough disk containing B and n is a large enough positive integer, then $\mathcal{S} = f^n(U)$ is foliated by uu -disks. More precisely, $\mathcal{S} = \bigcup_{t \in [0,1]} \mathcal{S}_t$ and \mathcal{S}_t is a uu -disk.

We could have assumed that B is a point of quadratic homoclinic tangency. Hence, to finish we need to analyze two cases.

In the first one, see figure 1, we can unfold the homoclinic tangency to obtain t_1 and t_2 such that $\mathcal{S}_{t_1} \cap W_{loc}^s(p, f_2)$ and $\mathcal{S}_{t_2} \cap W_{loc}^s(p, f_2)$ are non empty, and \mathcal{S}_t , $t_1 < t < t_2$ is a uu -disk in between of p and q . Therefore, $\tilde{\mathcal{S}} = \bigcup_{t \in [t_1, t_2]} \mathcal{S}_t$ is a folded manifold inside the unstable manifold as we wanted.

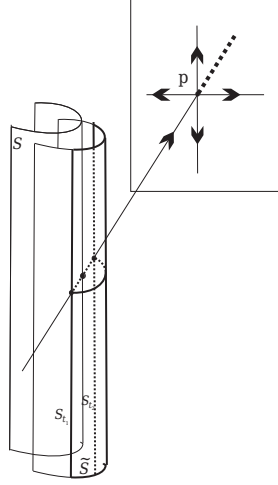


FIGURE 1. Folded manifold: first case

In the second case, replacing \mathcal{S} by a positive iterated, should exist t_1 and t_2 such that $\mathcal{S}_{t_1} \cap W_{loc}^s(q, f_2)$ and $\mathcal{S}_{t_2} \cap W_{loc}^s(q, f_2)$ are non empty intersections. Finally, unfolding the tangency as before, we also obtain a folded manifold. See figure 2. \square

4. NON EXISTENCE OF SYMBOLIC EXTENSIONS

In this section we prove Propostion D, and at the end we prove Theorem A.

Proof of Proposition D:

We give the proof only for volume preserving diffeomorphisms. The general case is completely similar.

Let f be a C^1 volume preserving diffeomorphism and $\mathcal{U} \subset \text{Diff}_m^1(M)$ be a neighborhood of f as in the assumptions.

By Robinson [33], Kupka-Smale's result is still true in the volume preserving setting. Hence, there exists an open and dense subset $\mathcal{A}_n \subset \mathcal{U}$ of diffeomorphisms

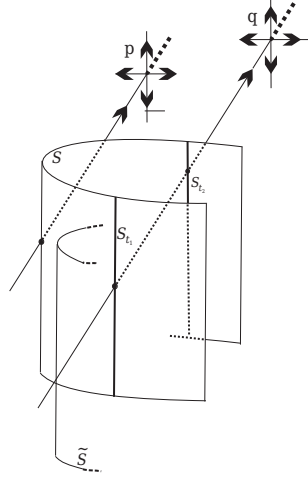


FIGURE 2. Folded manifold: second case

g such that every periodic point of g with period smaller than or equal to n is hyperbolic (or elliptic if M is a surface). Since the proposition for area preserving diffeomorphisms is contained in the Downarowicz-Newhouse's result [20], to avoid elliptic periodic points we suppose $\dim M \geq 3$.

We set $\mathcal{R}_1 = \cap \mathcal{A}_n$ which is a residual subset in \mathcal{U} . Let $\mathcal{R}_{1,m,n}$ be the open set of diffeomorphisms g in \mathcal{A}_n where m is the smallest one such that $\text{Per}_m(g) \neq \emptyset$. Hence,

$$\mathcal{A}_n = \cup_{j=1}^n \mathcal{R}_{1,j,n}.$$

We say that an increasing sequence of finite partitions (α_n) over M is *essential* for a diffeomorphism g if

1. $\text{diam}(\alpha_k) \rightarrow 0$ when $k \rightarrow \infty$, and
2. $\mu(\partial\alpha_k) = 0$ for every $\mu \in \mathcal{M}(g)$. Here, $\partial\alpha_k$ denotes the union of boundaries of all elements of the partition α_k .

By Proposition 4.1 in [20], we can assume there exists an increase sequence of partitions (α_n) over M and a residual subset $\mathcal{R}_2 \subset \mathcal{U}$ such that (α_n) is an essential sequence for every diffeomorphism $g \in \mathcal{R}_2$. From now on we fix this partition.

Recall, a f -invariant periodic set Λ with basis Λ_1 is *subordinated* to a finite partition α if for each positive integer n , there exists an element $B_n \in \alpha$ such that $f^n(\Lambda_1) \subset B_n$. In particular, if Λ is subordinated to α and $\mu \in \mathcal{M}(f|_\Lambda)$ then $h_\mu(\alpha) = 0$.

Let Λ be the hyperbolic basic set which exhibits robust homoclinic tangency for f . For every $g \in \mathcal{U}$, we denote by $H(\Lambda(g))$ the set of hyperbolic periodic points of g homoclinically related with $\Lambda(g)$, and we set

$$H_n(\Lambda(g)) = H(\Lambda(g)) \cap \text{Per}_n(g).$$

If p is a hyperbolic periodic point of g , $|\mu(p, g)| < 1 < |\lambda(p, g)|$ denote the two eigenvalues of $Dg^{\tau(p, g)}$ nearest one, i.e., if ν is an eigenvalue of $Dg^{\tau(p, g)}$ then either

$|\nu| \leq |\mu(p, g)|$ or $|\nu| \geq |\lambda(p, g)|$. We define

$$\chi(p, g) = \frac{1}{\tau(p, g)} \log \min\{|\lambda(p, g)|, |\mu(p, g)|^{-1}\},$$

for every hyperbolic periodic point p of g .

Finally, for any positive integer n , we say that a diffeomorphism $g \in \mathcal{U}$ satisfies property \mathcal{S}_n if for every $p \in H_n(\Lambda(g))$

1. *There exists a hyperbolic basic set of zero dimension $\Lambda(p, n)$ for g such that*

$$\Lambda(p, n) \cap \partial\alpha_n = \emptyset \text{ and } \Lambda(p, n) \text{ is subordinate to } \alpha_n.$$

3. *There exists an ergodic measure $\mu \in \mathcal{M}(f|_{\Lambda(p, n)})$ such that*

$$h_\mu(g) > \chi(p, g) - \frac{1}{n}.$$

4. *For every ergodic measure $\mu \in \mathcal{M}(f|_{\Lambda(p, n)})$, we have*

$$\rho(\mu, \mu_p) < \frac{1}{n},$$

where μ_p is the dirac measure on the orbit of p .

5. *For every periodic point $q \in \Lambda(p, n)$, we have*

$$\chi(q, g) > \chi(p, g) - \frac{1}{n}.$$

For positive integers $m \leq n$, let $\mathcal{D}_{m, n} \subset \mathcal{R}_{1, m, n}$ be the subset of diffeomorphisms satisfying property \mathcal{S}_n .

Claim: $\mathcal{D}_{m, n}$ is open and dense in $\mathcal{R}_{1, m, n}$.

This Claim is a conservative version of Lemma 3.3 in [14], which is an extension for symplectic diffeomorphisms of Lemma 5.1 in [20]. By Claim the proof is similar in spirit to Theorem 1.3 in [20]. See also [14]. However, just for sake of completeness we will give a sketch here.

We remark first if (α_n) is an essential sequence of partitions for f , then for any k fixed we set

$$h_k(\mu) = h_\mu(\alpha_k),$$

which is an infimum of continuous functions over $\mathcal{M}(f)$.

The following proposition relates the entropy structure of a diffeomorphism and non existence of symbolic extensions. It was also proved in [20].

Lemma 4.1. *Let $f \in \mathcal{R}_2$. If there exists a compact subset $\mathcal{E} \subset \mathcal{M}(f)$ and $\rho > 0$ such that*

$$\limsup_{\nu \rightarrow \mu, \nu \in \mathcal{E}} h_\nu(f) - h_k(\nu) \geq \rho, \text{ for every } \mu \in \mathcal{E} \text{ and } k \geq 0,$$

then f has no symbolic extension.

We set $\mathcal{R} = \bigcap_{n \geq 0} \bigcup_{m=0}^n \mathcal{D}_{m, n} \cap \mathcal{R}_2$, which is a residual subset in \mathcal{U} by Claim.

Now, let $f \in \mathcal{R}$ and define $\chi(f) = \sup\{\chi(p, f), p \in H(\Lambda(f))\}$. We denote by \mathcal{E} the closure of

$$\mathcal{E}_1 = \{\mu_p; p \in H(\Lambda(f)) \text{ and } \chi(p, f) > \chi(f)/2\}.$$

Since $f \in \mathcal{R}$, for any periodic point p such that $\mu_p \in \mathcal{E}_1$, it follows there exist ergodic measures $\nu_n \rightarrow \mu_p$ such that $h_{\nu_n}(f) > \chi(f)/2$. Moreover, $\nu_n \in \mathcal{M}(f|_{\Lambda(p, n)})$,

by Sigmund [34], ν_n is approximated by hyperbolic periodic measures also supported in the hyperbolic set $\Lambda(p, n)$, which by item 4 of property S_n should be in \mathcal{E}_1 . Hence, $\nu_n \in \mathcal{E}$ for every n .

Therefore, defining $\rho = \chi(f)/2 > 0$ and \mathcal{E} as before, by Lemma 4.1 f has no symbolic extensions.

Proof of Claim:

As in the proof of Technical Proposition in [14], the procedure is to find Newhouse's snakes (see Remark 4.2 for a definition) after a perturbation of a diffeomorphism exhibiting a homoclinic tangency, to obtain from them nice hyperbolic sets satisfying the conditions in property S_n . However, to find a diffeomorphism satisfying property S_n it's necessary to have an argument to obtain Newhouse's snakes related to any arbitrary hyperbolic periodic point. To do this we will use robustness of homoclinic tangency.

We denote by Λ the hyperbolic set of f exhibiting robust homoclinic tangency.

Let $g \in \mathcal{A}_n$ be an arbitrary diffeomorphism. By definition of \mathcal{A}_n , there exists a small neighborhood \mathcal{V} of g where the cardinality of periodic points of period smaller than or equal to n is constant.

Let $p \in H_n(\Lambda(g))$. Since $\Lambda(g)$ has a robust homoclinic tangency and p is homoclinic related with $\Lambda(g)$, after a perturbation we can assume that g exhibits a homoclinic tangency for the hyperbolic periodic point p . By regularization result of Ávila [4] we can suppose that g is C^∞ and then using pasting lemma we can assume that $g^{\tau(p,g)} = Dg^{\tau(p,g)}$ in some neighborhood of p (in local coordinate), say V . That is, $g^{\tau(p,g)}|_{(V \cap g^{-\tau(p,g)}(V))}$ is linear. Hence, $W_{loc}^s(p, g)$ and $W_{loc}^u(p, g)$ coincide with stable and unstable directions restrict to V .

For simplicity we suppose p is a hyperbolic fixed point of g .

Let q be the point of homoclinic tangency between $W_{loc}^s(p, g)$ and $W^u(p, g)$, such that $q \in V$ and $g^{-1}(q) \notin V$. Hence, there is a small neighborhood $U \subset V$ of q such that $g^{-1}(U) \cap V = \emptyset$. We denote by D the connected component of $W^u(p, g) \cap U$ that contains q . For convenience we suppose $\dim(T_q W_{loc}^s(p, g) \cap T_q W^u(p, g)) = 1$, which can be done after a perturbation.

We look to U in a local coordinate, being q the zero of this chart, and we consider the following splitting of space $\mathbb{R}^d = T_q D \oplus T_q D^\perp$. Since $D \cap U$ is an open disc inside $W^u(p, g)$, it follows that D is a graph of a map $r : T_q D \rightarrow T_q D^\perp$, which is so regular as g . That is, $D = (x, r(x))$ is the graphic of a C^∞ map r . Moreover, such map r is such that $Dr(q)$ is zero. Defining $\phi(x, y) = (x, y - r(x))$, if U is small enough, this map is a C^∞ volume preserving diffeomorphism from U to $\phi(U)$, and is C^1 close to identity in a small enough neighborhood of q . Therefore, by pasting lemma we can find a C^∞ volume preserving diffeomorphism h on M , C^1 -close to identity such that $h = \phi$ in some small neighborhood of q , and $h = Id$ outside U .

We define $g_1 = h \circ g$ which is a C^1 perturbation of g . Note, g_1 is such that $T_q D \cap U \subset W^u(p, g_1)$. Since $g^{-1}(U) \cap V = \emptyset$, it follows that $g_1 = g$ in V and so $g_1|_V$ is still linear, which implies $W_{loc}^s(p, g_1) \cap U = W_{loc}^s(p, g) \cap U = E^s(p, g) \cap U$. Hence, provided q was a non-transversal homoclinic point of p for g (i.e., $T_q D \cap E^s(p, g)$ is non trivial), g_1 exhibits an interval of homoclinic tangency containing q .

Let I be this interval of homoclinic tangency. By a coordinate change in U , we can suppose that $W_{loc}^s(p, g_1) \cap U = E^s(p, g_1) \cap U \subset \mathbb{R}^s \times \{0\}^u$, and $I \subset$

$\{(x_1, 0, \dots, 0), -3a \leq x_1 \leq 3a\}$, for some $a > 0$ small enough. We are now considering the usual coordinates in \mathbb{R}^d .

Let N be any large positive integer. By the same method as in the construction of h , for any $\delta > 0$ small enough we can find a volume preserving diffeomorphism $\Theta : M \rightarrow M$, $\delta - C^1$ near Id , such that $\Theta = Id$ in the complement of $B(q, 2a)$ and

$$\Theta(x, y) = \left(x_1, \dots, x_s, y_1 + A \cos \frac{\pi x_1 N}{2a}, y_2, \dots, y_u \right), \text{ for } (x, y) \in B(0, a) \subset U,$$

for $A = \frac{2Ka\delta}{\pi N}$, where K is a constant depending only on the local coordinate on U .

We define $g_2 = \Theta \circ g_1$ which is $\delta - C^1$ close to g_1 and moreover $g_2 = g_1$ in the complement of $g_1^{-1}(U)$. Note, g_2 exhibits N transversal homoclinic points for p inside U . More precisely, these points belong to $g_2(g^{-1}(I))$.

Remark 4.2. This kind of perturbation is the so called Newhouse's snake.

We remark that g_2 depends on N , but by abuse of notation, we use the same letter g_2 for every N .

From now on we will look to V in local coordinate. Moreover, we assume $E_p^s = \mathbb{R}^s \times \{0\}^u$ and $E_p^u = \{0\}^s \times \mathbb{R}^u$, where E_p^s and E_p^u are the stable and unstable directions of p with dimensions s and u , respectively. Observe g_2 is linear in V since g_2 is equal to g_1 in V .

For any positive large integer t , we can define a small rectangle $D_t = D^s \times D_t^u$, where $D^s = W_{loc}^s(p, g_2) \cap U$ and D_t^u is a small disk in $\{(0, \dots, 0, y_1, \dots, y_n), y_i \in \mathbb{R}^+ \text{ and } |y_i| < A/4\}$, such that $g_2^t(D_t)$ is a disk $A/4 - C^1$ close to the connected component of $W^u(p, g_2) \cap U$ containing the N transversal homoclinic points of g_2 in $g_2(g^{-1}(I))$. We fix t as being the smallest possible one such that D_t is defined. It is clearly that t depends on N , and goes to infinity if N goes.

Since N is large, it follows that A is so small which implies D_t is such that $g_2(D_t) \cap D_t$ has N disjoint connected components. Moreover, taking N larger if necessary, the maximal invariant set in D_t for g_2^t

$$\tilde{\Lambda}(p, N) = \bigcap_{j \in \mathbb{Z}} g_2^{tj}(D_t)$$

is a hyperbolic set.

Let

$$\Lambda(p, N) = \bigcup_{0 \leq j \leq t} g_2^j(\tilde{\Lambda}(p, N))$$

be the hyperbolic periodic set of g_2 induced by $\tilde{\Lambda}(p, N)$.

Now, if n is an arbitrary large positive integer n , we can proceed in the same way as in the proof of technical proposition in [14] to find a large positive integer N , such that $\Lambda(p, N)(\tilde{g})$ satisfies all items of property S_n for every diffeomorphism \tilde{g} C^1 -close to g_2 .

Finally, since the cardinality of Per_n is finite and constant for diffeomorphisms in \mathcal{V} , we can repeat the same process finitely many times to obtain an open set C^1 -close to g of diffeomorphisms satisfying property S_n . □

We finish the paper with the proof of Theorem A.

Proof of Theorem A:

Since $\text{Diff}_m^1(M)$ is a separable space, it follows there exists an enumerable dense subset $\{f_1, \dots, f_n, \dots\}$ of diffeomorphisms in $\text{Diff}_m^1(M)$.

If f_i is not partial hyperbolic, then by Theorem A we can suppose f_i exhibits a robust homoclinic tangency, after a perturbation. Then, by Proposition D there exist a neighborhood \mathcal{U}_i of f_i and a residual subset $\mathcal{R}_i \subset \mathcal{U}_i$ such that every diffeomorphism $g \in \mathcal{R}_i$ has no symbolic extensions. Hence, \mathcal{R}_i contains an enumerable intersection of open and dense subset in \mathcal{U} , say $\mathcal{R}_i = \cap \tilde{\mathcal{B}}_n^i$. We define $\mathcal{B}_n^i = \tilde{\mathcal{B}}_n^i \cup (cl(\mathcal{U}_i))^c$, which is in fact an open and dense subset of $\text{Diff}_m^1(M)$.

If f_i is partial hyperbolic we define \mathcal{B}_n^i as being the set of partial hyperbolic diffeomorphisms and the ones that are not approximated by partial hyperbolic diffeomorphisms, which is also an open and dense subset of $\text{Diff}_m^1(M)$.

Hence, we define

$$\mathcal{R} = \bigcap_{i,n \in \mathbb{N}} \mathcal{B}_n^i,$$

which is a residual subset in $\text{Diff}_m^1(M)$. Finally, note by construction that if $f \in \mathcal{R}$ is not partial hyperbolic, then f has no symbolic extensions. Which proves the theorem. \square

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REFERENCES

1. F. Abdenur, C. Bonatti, S. Crovisier, L. Diaz and L. Wen. Periodic points and homoclinic classes. *Ergod. Th. and Dynam. Sys.* 27 (2007), 1-22.
2. A. Arbieto and T. Catalan. Hyperbolicity in the Volume Preserving Scenario *preprint* (2010) [arXiv:1004.1664](#).
3. A. Arbieto and C. Matheus. A pasting lemma and some applications for conservative systems, *Erg. Th. and Dynamic. Sys.* 27 (2007), 1399-1417.
4. A. Ávila. On the regularization of conservative maps. *Acta Math.* 205 (2010), no. 1, 57-18.
5. C. Bonatti and S. Crovisier. Recurrence et genericite. *Inv. Math.* 158 (2004), 33-104
6. C. Bonatti and L. Diaz. Persistent Nonhyperbolic Transitive Diffeomorphisms. *Annals of Mathematics*. Vol. 143, No. 2 (Mar., 1996), pp. 357-396.
7. C. Bonatti and L. Diaz. Abundance of C^1 -robust homoclinic tangencies. (2009) *to appear in Trans. Amer. Math. Soc.*
8. C. Bonatti, L. Diaz and E. Pujals. A C^1 -generic dichotomy for diffeomorphisms: Weak forms of hyperbolicity or infinitely many sinks or sources. *Annals of Math.* 158 (2003), pp. 355-418.
9. M Boyle, D. Fiebig and U. Fiebig, Residual entropy, conditional entropy, and subshift covers, *Forum Math.*, 14 (2002), 713-757.
10. D. Burguet, C^2 surface diffeomorphisms have symbolic extensions, *Inventiones Mathematicae* 186 (2011), p.191-236.
11. D. Burguet and T. Fisher, Symbolic extensions for partially hyperbolic dynamical systems with 2-dimensional center bundle. *preprint* (2011).
12. J. Buzzi, Intrinsic ergodicity for smooth interval maps, *Israel J. Math.*, 100 (1997), 125-161.
13. C. Carballo, C. Morales and M. Pacifico. Homoclinic classes for generic C^1 vector fields. *Ergod. Th. and Dynam. Sys.* 23 (2003), pp. 403-415.
14. T. Catalan and A. Tahzibi, A Lower bound for topological entropy of generic non Anosov diffeomorphisms, *preprint* (2010) [arXiv:1011.2441](#).

15. S. Crovisier. Birth of homoclinic intersections: a model for the central dynamics of partially hyperbolic systems. *Annals of Math.* 172 (2010), 1641-1677.
16. S. Crovisier and E. Pujals. Essential hyperbolicity and homoclinic bifurcations: a dichotomy phenomenon/mechanism for diffeomorphisms. Preprint (2010). [arXiv:1011.3836](#)
17. S. Crovisier, M. Sambarino and D. Yang. Partial Hyperbolicity and Homoclinic Tangencies. Preprint (2011). [arXiv:1103.0869](#)
18. L. Diaz and T. Fisher, Symbolic extensions for partially hyperbolic diffeomorphisms. *Discrete Contin. Dyn. Syst.* 29 (2011), no. 4, 1419-1441.
19. L. Diaz, T. Fisher, M. Pacifico, and J. Vieitez, Entropy-expansiveness for partially hyperbolic diffeomorphisms, *preprint* (2010) [arXiv:1010.0721](#).
20. T. Downarowicz and S. E. Newhouse, Symbolic extension and smooth dynamical systems, *Inventiones Mathematicae*, 160 (2005), 453-499.
21. J. Franks. Necessary conditions for stability of diffeomorphisms. *Trans. A.M.S.* 158 (1971), 301-308.
22. S. Hayashi. Connecting Invariant Manifolds and the Solution of the C^1 Stability and C^1 -Stability Conjectures for Flows. *The Annals of Mathematics, Second Series*, Vol. 145, No. 1 (Jan., 1997), pp. 81-137.
23. F. Rodrigues Hertz, M.A. Rodrigues Hertz, A. Tahzibi and R. Ures. Creation of Blenders in the conservative setting. *Nonlinearity* 23 (2010), no. 2, 211- 223.
24. M. Hirsch, C. Pugh, M. Shub, Invariant manifolds. *Bull. AMS* 76 (1970), 1015-1019.
25. C. Liang, G. Liu, and W. Sun. Equivalent Conditions of Dominated Splittings for Volume-Preserving Diffeomorphism. *Acta Math. Sinica* 23 (2007), 1563-1576.
26. G. Liao, J. Yang and M. Viana, Entropy of diffeomorphisms away from tangencies , *Private communication*.
27. Mañé M. An Ergodic Closing Lemma. *The Annals of Mathematics 2nd Ser.*, Vol 116, No. 3. (Nov., 1982), 503-540.
28. S. E. Newhouse, Topological entropy and Hausdorff dimension for area preserving diffeomorphisms of surfaces, *Société Mathématique de France, Astérisque*, 51 (1978), 323-334.
29. S. E. Newhouse, Quasi-elliptic periodic points in conservative dynamical systems, *American Journal of Mathematics*, 99, No. 5 (1975), 1061-1087.
30. J. Palis. Global perspective for non-conservative dynamics. *Annales I. H. Poincaré - Analyse Non Linéaire*, v. 22. (2005), 485-507.
31. R. Potrie, Generic Bi-Lyapunov stable homoclinic classes. (2010) *Preprint*.
32. E. Pujals and M. Sambarino. Homoclinic tangencies and hyperbolicity for surface diffeomorphisms. *Annals of Mathematics*, 151 (2000), 961-1023.
33. R. C. Robinson. Generic properties of conservative systems. *Amer. J. Math.* 92 1970 562-603.
34. K. Sigmund, Generic Properties of Invariant Measures for Axiom A-Diffeomorphisms, *Invent. Math.* 11 (1970), 99-109.
35. Xia, Z. Homoclinic points in symplectic and Volume-Preserving diffeomorphisms, *Communications in Mathematical Physics*, 177 (1996), 435-449.
36. L. Wen. Homoclinic tangencies and dominated splittings. *Nonlinearity*, 15, 1445-1469 (2002).
37. Z. Xia e L. Wen, C^1 connecting lemmas, *Trans. A.M.S.*, Vol. 352, No. 11 (2000) pp. 5213-5230.

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